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LETTER TO THE EDITOR

Current localization in nonlinear inhomogeneous media

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Abstract. We consider the transport of electric current in inhomogeneous media with nonlinear conductivity. The interplay between disorder and nonlinearity gives rise to local enhancements of the current density. We study this current localization for a network of nonlinear resistors with the degree of nonlinearity varying at random through the system, which leads to a crossover from weak to strong localization with increasing voltage. We study the average fluctuations and the maximum value of the local current and discuss experimental consequences of these results.

Inhomogeneous media showing nonlinear response to external fields have been studied by various authors [1]. Most of these works focus on the global nonlinear response of the system (e.g. on nonlinear susceptibilities or conductances [2]). However, the combination of disorder and nonlinearity strongly affects the local behaviour and gives rise to novel phenomena. One of them is the voltage-driven crossover from a nearly homogeneous current density to a strongly inhomogeneous density, which is the subject of this letter.

Our interest in this transition is stimulated by recent observations of current localization in varistor ceramics [3, 4]. Varistors display highly nonlinear current–voltage characteristics which can be described by a power law in the breakdown region. This nonohmic behaviour is due to the existence of double Schottky barriers at grain boundaries [4]. Each grain boundary acts as a nonlinear resistor, with a nonlinearity exponent differing from one grain boundary to another and some boundaries even being linear resistors [5]. At low external voltages, the current is almost homogeneously distributed over the sample, whereas it becomes more and more localized with increasing voltage. This local enhancement of current eventually causes the failure of the varistor [6]. Therefore, the investigation of current localization is not only of academic interest but of practical importance as well.

In this letter, we study general properties of current localization on the example of a regular lattice whose bonds have a nonlinear conductivity $\sigma \sim V^{\alpha}$ with the exponent α being a random quantity. Random resistor networks (RRNs) have been widely used to model the transport behaviour in disordered systems [7, 8] or to simulate electrical and mechanical breakdown processes [9]. The percolation problem of linear RRNs has been studied [8], and the current distribution has been characterized [10]. Nonlinear RRNs whose bonds obey a power law relation between voltage and current with the exponent being constant throughout the network have been considered [11, 12].

However, in the physical systems considered here, e.g. varistor ceramics, the degree of nonlinearity varies throughout the material. Therefore the behaviour changes from that of a nearly homogeneous network at low voltages to that of a strongly disordered network

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for higher voltages, the current localization increasing with voltage. Only in the limit of very high voltages is the transport restricted to a single backbone and can be described by percolation theory. In this letter, we study a model with varying degree of nonlinearity which displays this voltage-driven current localization.

Our calculations are carried out on two-dimensional lattices with N nodes in both horizontal and vertical directions, and linear system sizes up to $N = 150$. The bonds of the network are assigned conductances

$$\sigma_{ij} = \left(V_{ij}\right)^{\alpha_{ij}} \tag{1}$$

where V_{ij} is the voltage drop over the bond connecting the nodes i and j and α_{ij} is a random variable. If the voltage drop is equal to one for all bonds, the system behaves like a network of identical ohmic resistors with unit conductance. In real systems, the conductances for unit voltage would certainly be random quantities, causing an additional inhomogeneity of the local currents. However, this inhomogeneity does not change with increasing voltage and is, at least at high voltages, small compared to that caused by the varying degree of nonlinearity. Thus we neglect it to keep the model simple.

The external voltage is applied to the top and bottom of the network. Periodic boundary conditions are used for the other two edges. The current flow through the network at fixed applied voltage is calculated using Kirchoff's law of current conservation at each node

$$\sum_j I_{ij} = \sum_j V_{ij}^{\alpha_{ij}} V_{ij} = 0 \qquad i = 1, \ldots, N^2 \tag{2}$$

where the sum runs over all neighbours j of the node i. Solutions of this set of N^2 nonlinear coupled equations were obtained numerically using a Newton–Raphson method. From the values of the voltages at each node, the local currents and the global current–voltage characteristic were computed. In this letter we present the results of simulations on a triangular lattice (see figure 1). Since the phenomenon of current localization does not depend on the particular distribution of α, a uniform distribution of α between zero and $\alpha_{\max} = 1$ is used [13].

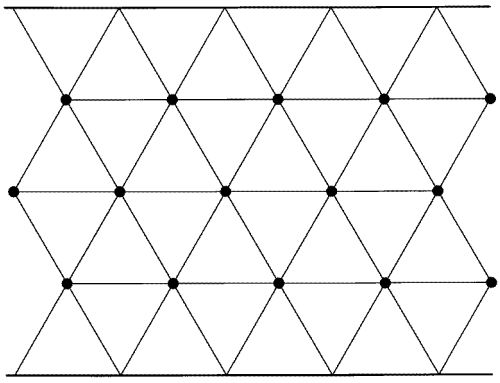

Figure 1. A schematic diagram of a triangular lattice with 5×3 nodes. The external voltage is applied between the horizontal bus bars at the top and bottom of the network.

We now turn to the discussion of the current localization. If V, denoting the external voltage normalized by the linear system size, is equal to unity, all sloped (i.e. not horizontal, see figure 1) bonds have unit conductance and contribute equally to the current flow through the network. No current is flowing through the horizontal bonds. As the voltage is increased,

the differences between the exponents α give rise to different conductances of the bonds. The current starts to seek out higher-conducting paths (large α) and begins to localize. The same happens if the voltage is reduced below $V = 1$, in which case the highly conducting bonds are those with the lowest values of α [14]. Figure 2 shows an example of this voltage-dependent localization.

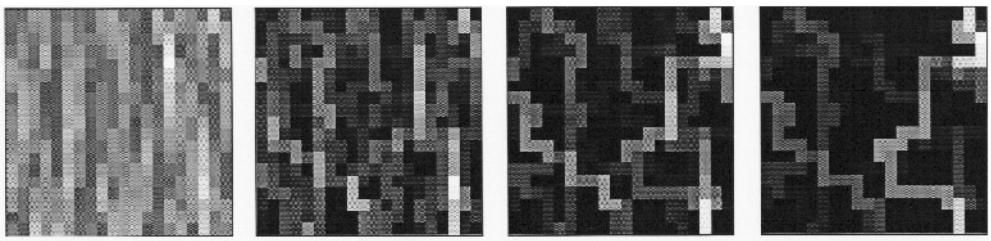

Figure 2. Evolution of the distribution of local currents with increasing applied voltage ($V = 4$, 1024, 4×10^6 and 1.7×10^{10} from left to right) for a 20×20 square lattice. Dark regions correspond to low, light regions to high current. The current distribution is rescaled to the maximum and minimum value in each frame.

We introduce two quantities to characterize the current localization. The first one is the ratio of the total current flowing through the network to the maximum current in a single bond,

$$n_{eff} = I_{total} / \max\left(I_{ij}\right) \tag{3}$$

with I_{ij} from (2). The quantity n_{eff} is related to the infinite moment of the current distribution and can be understood as being the effective number of current paths through the network. In the limit of a uniform system, the currents through the sloped bonds are all equal to the $2N$th part of the total current. Thus, $n_{eff} = 2N$. In the opposite limit of maximum localization, the current is flowing in just one filament. The maximum local current is therefore equal to the total current, and $n_{eff} = 1$. Although in these two extreme cases n_{eff} is an integer, in general it will not be. Figure 3 shows the effective path number for a single system with 100×100 nodes against normalized voltage.

A second quantity we use is related to the second moment of the current distribution. By analogy to the theory of localized electronic states in disordered systems [15], we define a participation ratio

$$p = \left\langle I_{ij} \right\rangle^2 \Big/ \left\langle I_{ij}^2 \right\rangle \tag{4}$$

where the average has to be carried out over all sloped bonds in the network. The participation ratio corresponds to the portion of bonds effectively participating in the current transport. For a system of identical bonds, all local currents are identical. Thus, $\langle I_{ij} \rangle^2 = \langle I_{ij}^2 \rangle$ and $p = 1$. In the opposite limit of localization to one filament, the N bonds belonging to the filament carry all the current and $p = 1/2N$. The participation ratio is shown in figure 4 for the 100×100 system. For $V = 1$, the system is composed of identical conductors and the participation ratio is unity; it decreases due to the current localization for higher or lower voltages.

A comparison of figures 3 and 4 shows that, although both quantities have their maxima for $V = 1$, they display an entirely different behaviour. The effective path number decays extremely fast as V deviates from unity while the participation ratio is still close to one. This is due to the fact that n_{eff}, being related to the infinite moment of the current distribution, is much more sensitive to fluctuations of the conductances than p, which is related to the

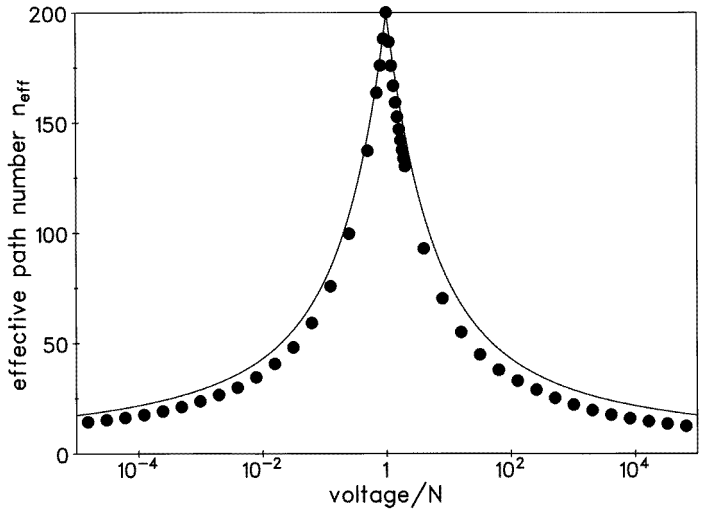

Figure 3. The effective number n_{eff} of current paths: dots, simulation data for a single sample; solid line, approximation (6).

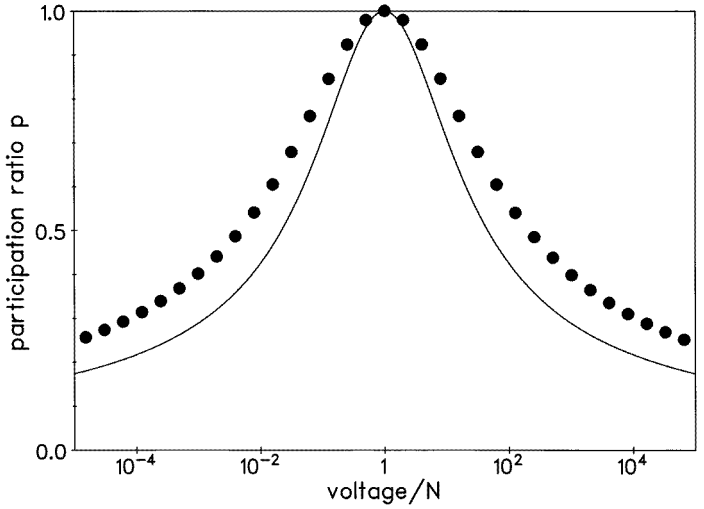

Figure 4. The portion p of bonds participating in the current transport: dots, simulation data for a single sample; solid line, approximation (7).

second moment and thus characterizes rather the overall behaviour of the system. However, in experiments the maximum local current, characterized by n_{eff}, is often very important. In varistor ceramics, e.g., it is responsible for the failure of the material even if the average current is not very high.

The different behaviour of n_{eff} and p can be understood by using a simple approximation which consists in looking at a single bond. This is justified for $V \approx 1$, where the localization is only weak so that hardly any redistribution of current through the horizontal bonds takes place, and the voltage drops across the sloped bonds differ only slightly. If we neglect these differences in the bond voltages, we can approximate the mean

value of the current and of the current squared using

$$\left\langle I_{ij}^{n}\right\rangle = \int_0^1 \mathrm{d}\alpha\, V^{n\alpha} V^{n} = \frac{V^n}{n \ln V}\left(V^n - 1\right). \tag{5}$$

For the triangular lattice, the total current is equal to $2N\langle I_{ij}\rangle$. For $V > 1$, the maximum current through a bond is given by the value at $\alpha = 1$, $I_{max} = V^2$, for $V < 1$ by the value at $\alpha = 0$, $I_{max} = V$. With (3), we obtain

$$\begin{aligned} n_{eff} &= (2N/\ln V)(V-1)/V \qquad V > 1 \\ n_{eff} &= (2N/\ln V)(V-1) \qquad\quad V < 1. \end{aligned} \tag{6}$$

Equation (6) is shown as the solid curve in figure 3. Although the approximation we used is very simple, it is in reasonable agreement with the numerical data close to $V = 1$; in particular it is able to reproduce the cusplike behaviour. Using (5) with the definition (4) of the participation ratio, we obtain

$$p = (2/\ln V)(V-1)/(V+1) \qquad V \approx 1. \tag{7}$$

Equation (7) corresponds to the solid curve in figure 4. It displays a horizontal tangent at $V = 1$. The deviations of the estimates (6) and (7) from the simulation data arise since collective effects of the whole array have been neglected. They will lead to smaller relative fluctuations and thus higher p as well as to rare combinations of highly conducting bonds which increase the maximum local current and thus decrease n_{eff}.

Another important question is whether the characteristics of the network depend on the details of the specific sample or are self-averaging for system size $N \to \infty$ and $V \to \infty$. Here it is necessary to distinguish two different regimes. If the limit $V \to \infty$ is performed first ($V \to \infty$ for a large, but finite system), the behaviour eventually reaches a critical percolation regime, where the current transport takes place on a single backbone. The physical properties strongly depend on the specific sample and their distributions remain very wide even for large systems. If, however, the thermodynamic limit ($N \to \infty$ for any finite V) is performed first, the system stays in a 'bulk' regime where a macroscopic number of paths is carrying the current. Hence the properties are self-averaging with their fluctuations vanishing in the thermodynamic limit. A detailed analysis of the system size dependence and the crossover between these two limiting regimes will be published elsewhere [16].

Evidence for the existence of the two regimes is also found by considering the effective conductance σ of the network, being defined as the total current divided by applied voltage. It is shown in figure 5 for a network with 100×100 nodes as a function of the normalized external voltage V. The asymptotic behaviour of σ for very low and very high voltages can be explained by percolation theory [8, 17], giving

$$\begin{aligned} \sigma &\sim V^{p_c \alpha_{\max}} \qquad V \ll 1 \\ \sigma &\sim V^{(1-p_c)\alpha_{\max}} \qquad V \gg 1 \end{aligned} \tag{8}$$

for low and high voltages, respectively. For the triangular lattice, $p_c = 0.347$. Equations (8) correspond to the solid lines in figure 5, giving a very good agreement with the numerical data in the low- and high-voltage regions, respectively, indicating that indeed a percolating backbone is responsible for the current transport in these limits. For the region around $V = 1$, the simulation data agree with the single bond approximation (5) for $n = 1$ (the inset of figure 5).

In conclusion, we have studied voltage-driven current localization in inhomogeneous media with nonlinear conductivity, using a random resistor network model where the degree

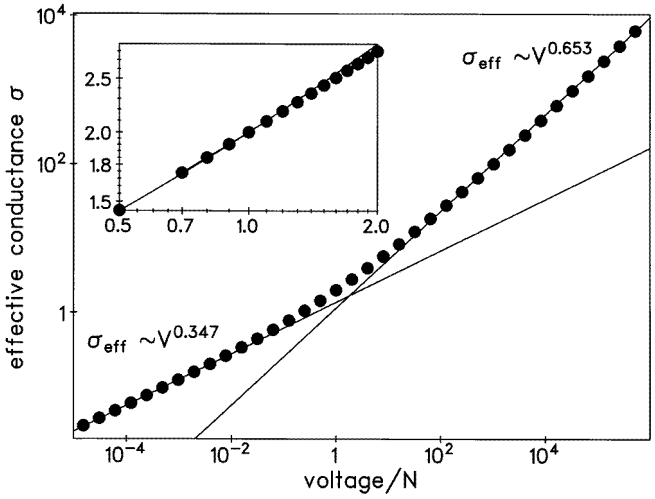

Figure 5. Effective conductance σ against normalized voltage: dots, simulation data for a single sample; solid lines, (8). Inset, the transition region around $V = 1$. The solid line corresponds to the approximation (5).

of nonlinearity varies throughout the system. We have found that the interplay between nonlinearity and disorder leads to a crossover from weak to strong localization which is not observed in previous models. In order to characterize the degree of current localization, two quantities have been introduced, which can be regarded as the effective number of current paths in the network and as the portion of bonds participating in the current transport. It turns out that the effective path number decreases extremely fast from its maximum value, indicating that the maximum local current is very high even if the average fluctuations of local current are still small, i.e. the participation ratio is close to one. The effective path number is very sensitive to rare events. Thus, a current-sensitive material will fail well before the average current reaches the critical value [6]. A detailed analysis of the current distribution function as well as results for different lattice types will be presented elsewhere [16].

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